

LAW OF THE ITERATED LOGARITHM FOR NONCOMMUTATIVE MARTINGALES

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ABSTRACT. We prove Kolmogorov's law of the iterated logarithm for quantum (noncommutative) martingales. The commutative case was due to Stout. The key ingredient is an exponential inequality proved recently by Junge and the author.

1. INTRODUCTION

In probability theory, law of the iterated logarithm (LIL) is among the most important limiting theorems and has been studied extensively in different contexts. The early contributions in this direction for independent increments were made by Khintchine, Kolmogorov, Hartman–Wintner, etc; see [1] for more history of this subject. The martingale version of Kolmogorov's LIL was due to Stout [15]. Hartman–Wintner's LIL requires the sequence to be independent and identically distributed (i.i.d.) while Kolmogorov's version only assumes the sequence to be independent with a suitable growing condition. In the last decade, there has been new development for LIL results of dependent random variables; see [18, 19] and the references therein for more details. However, it seems that the LIL in quantum (= noncommutative) probability theory has only been proved recently by Konwerska [10, 11] for Hartman–Wintner's version. The goal of this paper is to prove Kolmogorov's version of LIL for noncommutative martingales.

Let us first recall the LIL results due to Komogorov and Hartman–Wintner. Let $(Y_n)_{n \in \mathbb{N}}$ be an independent sequence of square-integrable, centered, real random variables. Put $S_n = \sum_{i=1}^n Y_i$ and $s_n^2 = \text{Var}(S_n) = \sum_{i=1}^n \mathbb{E}(Y_i^2)$. Here and in the following \mathbb{E} denotes the expectation operator and Var denotes the variance. For any $x > 0$, we define the notation $L(x) = 1 \vee \ln \ln x$. In 1929, Kolmogorov proved that if $s_n^2 \rightarrow \infty$

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and

$$(1) \quad |Y_n| \leq \alpha_n \frac{S_n}{\sqrt{L(s_n^2)}} \text{ a.s.}$$

for some positive sequence (α_n) such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, then

$$(2) \quad \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{s_n^2 L(s_n^2)}} = \sqrt{2} \text{ a.s..}$$

Later on, Hartman–Wintner [4] proved that if (X_n) is an i.i.d. sequence of real, centered square-integrable random variables with variance $\text{Var}(X_i) = \sigma^2$, then

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{nL(n)}} = \sqrt{2}\sigma \text{ a.s..}$$

de Acosta [2] simplified the proof of Hartman–Wintner. To compare the two results, if the sequence (Y_n) are i.i.d. and uniformly bounded, then the two results coincide. Apparently, Hartman–Wintner’s LIL does not contain Kolmogorov’s version as a special case. However, Kolmogorov’s LIL can be used in a truncation procedure to prove other LIL results; see, e.g., [14].

Komogorov’s LIL was generalized to martingale sequences by Stout [15]. Let $(X_n, \mathcal{F}_n)_{n \geq 1}$ be a martingale sequence with $\mathbb{E}(X_n) = 0$. Let $Y_n = X_n - X_{n-1}$ for $n \geq 1$, $X_0 = 0$ be the associated martingale differences. Put $s_n^2 = \sum_{i=1}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}]$. Then Stout proved that if $s_n^2 \rightarrow \infty$ and (1) holds, then $\limsup_{n \rightarrow \infty} X_n / \sqrt{s_n L(s_n)} = \sqrt{2}$ a.s.; see [15].

To state our main results, let us set up the noncommutative framework. Throughout this paper, we consider a noncommutative probability space (\mathcal{N}, τ) . Here \mathcal{N} is a finite von Neumann algebra and τ a normal faithful tracial state, i.e. $\tau(xy) = \tau(yx)$ for $x, y \in \mathcal{N}$. For $1 \leq p < \infty$, define $\|x\|_p = [\tau(|x|^p)]^{1/p}$ and $\|x\|_\infty = \|x\|$ for $x \in \mathcal{N}$. In this paper $\|\cdot\|$ will always denote the operator norm. Then the noncommutative L_p space $L_p(\mathcal{N}, \tau)$ (or $L_p(\mathcal{N})$ for short) is the completion of \mathcal{N} with respect to $\|\cdot\|_p$. τ -measurable operators affiliated to (\mathcal{N}, τ) are also called noncommutative random variables; see [3, 16] for more details on the measurability and noncommutative L_p spaces. Let $(\mathcal{N}_k)_{k=1,2,\dots} \subset \mathcal{N}$ be a filtration of von Neumann subalgebras with conditional expectation $E_k : \mathcal{N} \rightarrow \mathcal{N}_k$. Then $E_k(1) = 1$ and $E_k(axb) = aE_k(x)b$ for $a, b \in \mathcal{N}_k$ and $x \in \mathcal{N}$. It is well known that E_k extends to contractions on $L_p(\mathcal{N}, \tau)$ for $p \geq 1$; see [7].

Following [10], a sequence (x_n) of τ -measurable operators is said to be almost uniformly bounded by a constant $K \geq 0$, denoted by $\limsup_{n \rightarrow \infty} x_n \leq K$ *a.u.*, if for any $\varepsilon > 0$ and any $\delta > 0$, there exists a projection e with $\tau(1 - e) < \varepsilon$ such that

$$(3) \quad \limsup_{n \rightarrow \infty} \|x_n e\| \leq K + \delta;$$

and (x_n) is said to be bilaterally almost uniformly bounded by a constant $K \geq 0$, denoted by $\limsup_{n \rightarrow \infty} x_n \leq K$ *b.a.u.*, if (3) is replaced by

$$\limsup_{n \rightarrow \infty} \|e x_n e\| \leq K + \delta.$$

Clearly, $\limsup_{n \rightarrow \infty} x_n \leq K$ *a.u.* implies $\limsup_{n \rightarrow \infty} x_n \leq K$ *b.a.u.*

For a τ -measurable operator x and $t > 0$, the generalized singular numbers [3] are defined by

$$\mu_t(x) = \inf\{s > 0 : \tau(1_{(s, \infty)}(|x|)) \leq t\}.$$

In this paper, $1_{(s, \infty)}(|x|)$ denotes the spectral projection of $|x|$. According to [10], a sequence of operators (x_i) is said to be uniformly bounded in distribution by an operator y if there exists $K > 0$ such that $\sup_i \mu_t(x_i) \leq K \mu_{t/K}(y)$ for all $t > 0$. Let (x_n) be a sequence of mean zero self-adjoint independent random variables. Konwerska [11] proved that if (x_n) is uniformly bounded in distribution by a random variable y such that $\tau(|y|^2) = \sigma^2 < \infty$, then

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{nL(n)}} \sum_{i=1}^n x_i \leq C\sigma \quad \text{b.a.u.}$$

Note that if the sequence (x_n) is i.i.d., which is the case in the original version of Hartman–Wintner’s LIL, then (x_n) is uniformly bounded in distribution by x_1 . Essentially, the condition of uniform boundedness in distribution requires the sequence to be almost identically distributed.

Stout proved Hartman–Wintner’s LIL in [14] under the additional assumption that the martingale differences are stationary ergodic. At the time of this writing, it is still not clear to us whether a “genuine” version (i.e., it does not satisfy Kolmogorov’s growing condition) of Hartman–Wintner’s LIL is possible for noncommutative martingales. However, we are able to replace the uniform boundedness assumption by a growing condition for the martingale differences, and extend Stout’s result on Kolmogorov’s LIL to the setting of noncommutative martingales. Let $s_n^2 = \|\sum_{i=1}^n E_{i-1}(d_i^2)\|_\infty$ and $u_n = [L(s_n^2)]^{1/2}$.

Theorem 1.1. *Let $0 = x_0, x_1, x_2, \dots$ be a self-adjoint martingale sequence in (\mathcal{N}, τ) . Suppose $s_n^2 \rightarrow \infty$ and $\|d_n\|_\infty \leq \alpha_n s_n / u_n$ for constants $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Then*

$$\limsup_{n \rightarrow \infty} \frac{x_n}{s_n u_n} \leq 2 \text{ b.a.u.}$$

So far as we know, this is the first result on the LIL for noncommutative martingales. A natural question is to ask for the lower bound of LIL. As observed in [10], however, one can only expect an upper bound for LIL in the general noncommutative setting. Indeed, consider a free sequence of semicircular random variables (x_n) (the so-called free Gaussian random variables [17]) such that the law of x_n is $\gamma_{0,2}$ (in notation, $x_n \sim \gamma_{0,2}$). Here $\gamma_{0,2}$ has density function $p(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$ for $-2 \leq x \leq 2$. Then it is well known in free probability theory that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_{0,2}.$$

It follows that $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i / \sqrt{nL(n)} = 0$ since a random variable with law $\gamma_{0,2}$ is bounded. Therefore the lower bound does not hold. Comparing our LIL results with classical ones, we lose a constant of $\sqrt{2}$. However, since there is no hope to obtain a lower bound, we are more interested in the order of the fluctuation for general noncommutative martingales. It is also commonly acknowledged that going from the commutative case to the noncommutative setting usually requires considerably more technologies [13], and the martingale case is even more complicated than independent case. Due to these reasons, it seems fair to have the constant 2 in the noncommutative martingale setting.

We will recall some preliminary facts in Section 2. The main result will be proved in Section 3.

2. PRELIMINARIES

In this section, we give some basic definitions and collect (without proof) some preliminary facts. The main concern is to give the noncommutative analogues of classical results like almost sure convergence, Doob's inequality, Borel-Contelli lemma, etc. These facts have been used successfully to prove noncommutative ergodic theorems [8] and noncommutative LIL for the independent case [11].

Let us recall the vector valued noncommutative L_p spaces for $1 \leq p \leq \infty$ introduced by Pisier [12] and Junge [6]. Let (x_n) be a sequence

in $L_p(\mathcal{N})$ and define

$$\|(x_n)\|_{L_p(\ell_\infty)} = \inf\{\|a\|_{2p}\|b\|_{2p} : x_n = ay_nb, \|y_n\|_\infty \leq 1\}.$$

Then $L_p(\ell_\infty)$ is defined to be the closure of all sequences with $\|(x_n)\|_{L_p(\ell_\infty)} < \infty$. It was shown in [8] that if every x_n is self-adjoint, then

$$\|(x_n)\|_{L_p(\ell_\infty)} = \inf\{\|a\|_p : a \in L_p(\mathcal{N}), a \geq 0, -a \leq x_n \leq a \text{ for all } n \in \mathbb{N}\}.$$

Similarly, Junge and Xu introduced in [8] the space $L_p(\ell_\infty^c)$ with norm

$$\begin{aligned} & \|(x_i)_{i \in I}\|_{L_p(\ell_\infty^c)} \\ &= \inf\{\|a\|_p : a \in L_p(\mathcal{N}), a \geq 0, -a \leq x_i^* x_i \leq a \text{ for all } i \in I\} \\ &= \inf\{\|b\|_p : x_i = y_i b, \|y_i\|_\infty \leq 1 \text{ for all } i \in I\}. \end{aligned}$$

They also defined $L_p(c_0) \subset L_p(\ell_\infty)$ as the space of all $(x_n) \subset L_p(\mathcal{N})$ such that there are $a, b \in L_{2p}(\mathcal{N})$ and $(y_n) \subset \mathcal{N}$ satisfying

$$(4) \quad x_n = ay_nb \quad \text{and} \quad \lim_{n \rightarrow \infty} \|y_n\|_\infty = 0.$$

Konwerska [10] used the space $L_p(c) \subset L_p(\ell_\infty)$, which is defined similar to $L_p(c_0)$ with (4) replaced by

$$x_n = ay_nb \quad \text{and} \quad \lim_{n \rightarrow \infty} \|y_n - y_\infty\|_\infty = 0 \text{ for some } y_\infty \in \mathcal{N}.$$

The norms of $L_p(c_0)$ and $L_p(c)$ are the same as $L_p(\ell_\infty)$. Clearly $L_p(c_0) \subset L_p(c)$. It is easy to verify that all the spaces we mentioned above are Banach spaces.

The following result is the noncommutative version of Doob's maximal inequality proved by Junge [6].

Theorem 2.1. *Let $2 \leq p \leq \infty$. Then, for any $x \in L_p(\mathcal{N})$, there exists $b \in L_p(\mathcal{N})$ and a sequence of contractions $(y_n) \subset \mathcal{N}$ such that*

$$\|b\|_p \leq 2^{2/p} \|x\|_p \quad \text{and} \quad E_n x = y_n b, \text{ for all } n \geq 0.$$

Suppose $(x_i)_{m \leq i \leq n}$ is a martingale sequence in $L_p(\mathcal{N})$. According to Theorem 2.1, there exist $b \in L_p(\mathcal{N})$ and contractions $(y_i)_{m \leq i \leq n} \subset \mathcal{N}$ such that $x_i = y_i b$ for $m \leq i \leq n$ and $\|b\|_p \leq 2^{2/p} \|x_n\|_p$ for $p \geq 2$. It follows that

$$\|(x_i)_{m \leq i \leq n}\|_{L_p(\ell_\infty^c)} \leq 2^{2/p} \|x_n\|_p.$$

Doob's inequality will be used in this form in the proof of our main result.

Our proof of LIL for martingales relies on the following exponential inequality proved in [9]. Its proof was based on Oliveira's approach to the matrix martingales [5].

Lemma 2.2. *Let (x_k) be a self-adjoint martingale sequence with respect to the filtration (\mathcal{N}_k, E_k) and $d_k = x_k - x_{k-1}$ be the associated martingale differences such that*

$$\text{i) } \tau(x_k) = x_0 = 0; \text{ ii) } \|d_k\| \leq M; \text{ iii) } \sum_{k=1}^n E_{k-1}(d_k^2) \leq D^2 1.$$

Then

$$\tau(e^{\lambda x_n}) \leq \exp[(1 + \varepsilon)\lambda^2 D^2]$$

for all $\varepsilon \in (0, 1]$ and all $\lambda \in [0, \sqrt{\varepsilon}/(M + M\varepsilon)]$.

Another important tool in our proof is a noncommutative version of Borel-Contelli lemma. To state this result, we recall from [10] that for a self-adjoint sequence $(x_i)_{i \in I}$ of random variables, the column version of tail probability is by definition

$$\begin{aligned} & \text{Prob}_c\left(\sup_{i \in I} x_i > t\right) \\ &= \inf\{s > 0 : \exists \text{ a projection } e \text{ with } \tau(1 - e) < s \\ & \text{and } \|x_i e\|_\infty \leq t \text{ for all } i \in I\}. \end{aligned}$$

The following two lemmas are taken from [10].

Lemma 2.3 (Noncommutative Borel-Contelli lemma). *Let $\cup_n I_n = \{n \in \mathbb{N} : n \geq n_0\}$ for some $n_0 \in \mathbb{N}$ and (z_n) be a sequence of self-adjoint random variables. If for any $\delta > 0$,*

$$\sum_{n \geq n_0} \text{Prob}_c\left(\sup_{m \in I_n} z_m > \gamma + \delta\right) < \infty,$$

then

$$\limsup_{n \rightarrow \infty} z_n \underset{\text{a.u.}}{\leq} \gamma.$$

Using the notation Prob_c , we state a version of Chebyshev inequality.

Lemma 2.4. *Let $(x_i)_{i \in I}$ a self-adjoint sequence of random variables. For $t > 0$ and $1 \leq p < \infty$,*

$$\text{Prob}_c(\sup_n x_n \geq t) \leq t^{-p} \|x\|_{L_p(\ell_\infty^c)}^p.$$

3. LAW OF THE ITERATED LOGARITHM

According to [1], the original proof of Kolmogorov's LIL is comparably expensive as that of Hartman-Wintner. However, our proof of Kolmogorov's LIL here seems to be relatively easier than Hartman-Wintner's version for the commutative case due to the exponential inequality (Lemma 2.2).

Proof of Theorem 1.1. The proof is similar to the previous one. Thanks to the growing condition, we do not even need the truncation procedure. Let $\eta \in (1, 2)$ be a constant which we will determine later. Following the stopping rule in [15], we define $k_0 = 0$ and for $n \geq 1$,

$$k_n = \inf\{j \in \mathbb{N} : s_{j+1}^2 \geq \eta^{2n}\}.$$

Then $s_{k_n+1}^2 \geq \eta^{2n}$ and $s_{k_n}^2 < \eta^{2n}$. Note that given $\varepsilon' > 0$ there exists $N_1 > 0$ such that for $n > N_1$,

$$\begin{aligned} & s_{k_n+1}^2 u_{k_n+1}^2 / (s(k_{n+1})^2 u(k_{n+1})^2) \\ & \geq \eta^{-2} \ln \ln \eta^{2n} / \ln \ln \eta^{2(n+1)} \geq (1 - \varepsilon')^2 \eta^{-2}. \end{aligned}$$

Then $s_m u_m \geq (1 - \varepsilon') \eta^{-1} s(k_{n+1}) u(k_{n+1})$ for $k_n < m \leq k_{n+1}$. For any $\delta' > 0$, we can find $\delta > 0$ and $\eta \in (1, 2)$ such that $1 + \delta' > \eta(1 - \varepsilon')^{-1}(1 + \delta)$. Fix $\beta > 0$ which will be determined later. We then have

$$\begin{aligned} & \text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{x_m}{s_m u_m} > \beta(1 + \delta') \right) \\ & \leq \text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{\lambda x_m}{s(k_{n+1}) u(k_{n+1})} > \lambda \beta(1 + \delta) \right). \end{aligned}$$

By Lemma 2.4 and Lemma 2.1, we have for $p > 2$,

$$\begin{aligned} & \text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{\lambda x_m}{s(k_{n+1}) u(k_{n+1})} > \lambda \beta(1 + \delta) \right) \\ & \leq (\lambda \beta(1 + \delta))^{-p} \left\| \left(\frac{\lambda x_m}{s(k_{n+1}) u(k_{n+1})} \right)_{k_n < m \leq k_{n+1}} \right\|_{L_p(\ell_\infty^c)}^p \\ & \leq (\lambda \beta(1 + \delta))^{-p} (2^{2/p})^p \left\| \frac{\lambda x(k_{n+1})}{s(k_{n+1}) u(k_{n+1})} \right\|_p^p. \end{aligned}$$

Using the elementary inequality $|u|^p \leq p^p e^{-p}(e^u + e^{-u})$, functional calculus and Lemma 2.2 with $M = \alpha(k_{n+1}) s(k_{n+1}) / u(k_{n+1})$, $D^2 = s(k_{n+1})^2$, we find

$$\begin{aligned} & \left\| \frac{\lambda x(k_{n+1})}{s(k_{n+1}) u(k_{n+1})} \right\|_p^p \\ & \leq p^p e^{-p\tau} \left(\exp \left(\frac{\lambda x(k_{n+1})}{s(k_{n+1}) u(k_{n+1})} \right) + \exp \left(- \frac{\lambda x(k_{n+1})}{s(k_{n+1}) u(k_{n+1})} \right) \right) \\ & \leq 2 \left(\frac{p}{e} \right)^p \exp \left(\frac{(1 + \varepsilon) \lambda^2}{u(k_{n+1})^2} \right) \end{aligned}$$

provided $0 \leq \lambda \leq \frac{\sqrt{\varepsilon}u(k_{n+1})^2}{(1+\varepsilon)\alpha(k_{n+1})}$ and $0 < \varepsilon \leq 1$. Hence we obtain

$$\begin{aligned} & \text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} > \lambda\beta(1+\delta) \right) \\ & \leq 8 \left(\frac{p}{\lambda\beta(1+\delta)e} \right)^p \exp \left(\frac{(1+\varepsilon)\lambda^2}{u(k_{n+1})^2} \right). \end{aligned}$$

Now optimizing in p gives $p = \lambda\beta(1+\delta)$ and thus,

$$\begin{aligned} & \text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} > \lambda\beta(1+\delta) \right) \\ & \leq 8 \exp \left(\frac{(1+\varepsilon)\lambda^2}{u(k_{n+1})^2} - \beta(1+\delta)\lambda \right). \end{aligned}$$

Put $\lambda = \beta(1+\delta)u(k_{n+1})^2/(2(1+\varepsilon))$. Since $\alpha_n \rightarrow 0$, for any $\varepsilon > 0$ there exists $N_2 > 0$ such that for $n > N_2$, $0 < \alpha(k_{n+1}) \leq \frac{2\sqrt{\varepsilon}}{\beta(1+\delta)}$, which ensures that we can apply Lemma 2.2. This also implies $p > 2$. It follows that

$$\text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} > \lambda\beta(1+\delta) \right) \leq (\ln s(k_{n+1})^2)^{-\frac{\beta^2(1+\delta)^2}{4(1+\varepsilon)}}.$$

Notice that $s(k_{n+1})^2 \geq s(k_n+1)^2 \geq \eta^{2n}$. Setting $\beta = 2$ in the beginning of the proof, we have

$$\text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} > \lambda\beta(1+\delta) \right) \leq [(2 \ln \eta)n]^{-\frac{(1+\delta)^2}{1+\varepsilon}}.$$

By choosing ε small enough so that $(1+\delta)^2/(1+\varepsilon) > 1$, we find that for $n_0 = \max\{N_1, N_2\}$,

$$\sum_{n \geq n_0} \text{Prob}_c \left(\sup_{k_n < m \leq k_{n+1}} \frac{x_m}{s_m u_m} > \beta(1+\delta') \right) < \infty.$$

Then Lemma 2.3 gives the desired result. \square

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